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A nonlocal operator basis from the path representation of the $\mathcal{M}(k+1, k+2)$ and the $\mathcal{M}(k+1, 2k+3)$ minimal models*

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Abstract

We reinterpret a path describing a state in an irreducible module of the unitary minimal model $\mathcal{M}(k+1, k+2)$ in terms of a string of charged operators acting on the module's ground-state path. Each such operator acts non-locally on a path. The path characteristics are then translated into a set of conditions on sequences of operators that provide an operator basis. As an application, we re-derive the vacuum finite fermionic character by constructing the generating function of these basis states. These results generalize directly to the $\mathcal{M}(k+1, 2k+3)$ models, the close relatives of the unitary models in terms of path description.

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1. Introduction

1.1. General orientation

The minimal models [3] are solvable models that describe universality classes of two-dimensional critical phenomena. As such, they are physically quite important. Their exact solvability is rooted in the representation theory of the Virasoro algebra. Every field is in correspondence with a state and the space of states is a finite direct sum of highest-weight modules. In each such module there is an infinite number of singular vectors. These lead to differential equations for the correlation functions. This accounts for the models' exact solvability. Irreducible modules are obtained by factoring out these singular vectors using an inclusion–exclusion process that results in an expression of the character in the form of an infinite alternating sum (see e.g., [10]). Although these characters can be obtained in closed

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forms, the states in the irreducible modules do not have a clear physical interpretation in this description.

The discovery of fermionic character formulae [23, 24], which are positive multiple sums, suggests the existence of an alternative and more physical approach to the representation theory of the Virasoro algebra that would be based on the concept of quasi-particles. In this picture, the Hilbert space is constructed by a filling process subject to restriction rules. Along that line, a useful guide is provided by the realization of the minimal models as statistical models, in particular, the restricted-solid-on-solid (RSOS) models of [2, 15]. The solution of these models embodies a description of the states in terms of configuration sums (or paths) that are tailor-made for a reinterpretation in terms of quasi-particles. Once the quasi-particles are identified, the next step would be to interpret them in a conformal-field-theoretical setting.

The present work lies mid-way within this program. Roughly, we abstract the path description of a particular class of minimal models into an operator basis whose interpretation is yet to be formulated. This is clarified in the following subsections.

1.2. Unitary minimal models: paths and heuristic considerations underlying their operator interpretation

For the larger part of this work, we will be concerned with the unitary minimal models $\mathcal{M}(k+1, k+2)$ [16]. Their states have a particularly simple representation as lattice paths. It is inherited from the models' off-critical formulation in terms of the Andrews–Baxter–Forrester RSOS models in regime III [2, 17]. More precisely, the solution of this statistical model by the corner-transfer-matrix method induces a representation of every state in the form of a particular configuration. A configuration is the specification of the set of values of the order parameter, the height variable, say y , for all integral positions x between 0 and a certain length L . The height is bounded by the parameter k characterizing the model: $0 \leq y \leq k$ (where k is related to the parameter r of [2] by $r = k + 2$). Every configuration is uniquely specified by its contour obtained by linking adjacent heights. Since two adjacent heights can differ by ± 1 , the links are either North-East (NE) or South-East (SE) edges. This contour defines an integer-lattice path.

To each path, we associate a weight. The weight of a path is simply the sum of the weight of all its vertices in between 1 and $L - 1$; each vertex has weight $x/2$, where x is its horizontal position, unless it is a local maximum or minimum, in which case its weight is 0.

Classes of paths are specified by the boundary conditions, namely the pairs of values (y_0, y_L) . Quite remarkably, the (q -weighted) generating function for all paths with fixed boundary conditions becomes, in the limit where $L \rightarrow \infty$, is equal to the (normalized) character of the irreducible module $\chi_{r,s}(q)$, for some pair (r, s) determined by (y_0, y_L) , of the $\mathcal{M}(k+1, k+2)$ models [9]. Each path is thus a representation of a state in a specific unitary irreducible module and a finite path describes a state in a natural finitization of the module [26].

Characters can be read off this path representation in at least two different ways. We can derive recurrence relations on the finite-path generating function by considering the effect of removing an edge at the end of the path [2]. The immediate solution of these relations yields a finitized version of the Rocha–Caridi (or bosonic) character formulae. A different approach consists in considering the generating function for paths as the grand-canonical partition function for a one-dimensional gas of particles subject to fermionic-type exclusion rules. This interpretation leads to a positive multiple-sum expression of the characters [28].

A natural question is the following: is there a natural operator formulation underlying this fermi-gas representation?

As a first guiding observation, let us recall (cf [5, 9, 12, 28]) that the $\mathcal{M}(k+1, k+2)$ model is dual to the \mathcal{Z}_k parafermionic theory [31]. The duality is defined at the level of the weight function. As we said before, the vertices that contribute to the weight of a $\mathcal{M}(k+1, k+2)$ path are those which do not correspond to local extrema and their contribution is $x/2$. Paths in the dual model are defined as in the original model but they are weighted in a dual way: only the extrema do contribute and they contribute to the value $x/2$. Each path is in correspondence with a state of the finitized parafermionic module [20]. Since a character is a q -weighted sum over paths, the duality between the weight function is lifted, at the level of characters, to a $q \leftrightarrow q^{-1}$ transformation (up to easily fixed L -dependent powers of q).

Now in the parafermionic case, the path representation is very close to the quasi-particle description of the conformal model [18, 25]. As demonstrated in [20], each peak can be identified with the mode of a parafermionic operator whose parafermionic charge is twice the charge of the peak (to be defined below) and its mode index is minus the x position.

A path is a sequence of peaks and, as we just indicated, in the parafermionic context this sequence can be interpreted as a string of parafermionic modes. The path itself is manifestly independent upon the way it is weighted. It is thus tempting to guess that, loosely speaking, the operator formulation underlying the path representation of the minimal models could similarly be described by parafermionic-type operators but which would then, in the light of the $q \leftrightarrow q^{-1}$ duality, acts in some non-local way.

This sets the stage for the present investigation. We search for an operator description of the $\mathcal{M}(k+1, k+2)$ paths and expect this to be, somehow, non-local.

1.3. From paths to an operator basis

Concretely, we look for a description of an irreducible $\mathcal{M}(k+1, k+2)$ module (specified by some boundary conditions) in terms of a string of operators acting on the path with the lowest energy (or weight) within this module. We next lift the set of constraints fixed by the path on allowed sequences of such operators to an independent operator basis describing irreducible modules.

It turns out not to be difficult to work out an operator interpretation of the $\mathcal{M}(k+1, k+2)$ paths. It first consists in interpreting each weight-contributing vertex as the insertion point of an operator of energy equal to the weight of the vertex; if the vertex has coordinates (x, y) , this weight is $x/2$. Roughly, one type of operator acts at a peak position and creates a NE edge linking x and $x+1$. Let us call this operator b_x . By construction, two operators cannot act at the same point since after one action, a vertex that corresponds to a maximum is transformed into the starting point of a straight-up segment. This makes these operator of fermionic type ($b_x^2 = 0$). By creating a NE edge, b_x lifts the height of the whole tail of the path, from the point $x+1$ to its right end, by one unit. This is definitely a non-local action. It also clearly modifies the path boundary condition at the right extremity. The precise definition of this action is presented in section 2.

Because each action of a b operator lifts the path tail, a specific module cannot be described solely in terms of such b -type operators acting in all possible ways on the ground-state configuration of the module, since all paths pertaining to a given module must have the same end condition. One also needs to have an operator that decreases the height of the path tail by creating a SE edge. Let us call such an operator b^* . We thus see that the states in a given module (i.e., with fixed values of (y_0, y_L)) must be described by sequences containing equal numbers of b and b^* operators acting on the module's ground state. Moreover, since the height of a path can never be larger than k , there cannot be more than $k-1$ adjacent b or b^* operators. The full set of relations is presented in section 2.

In the present paper, we provide an application of this operator formalism, by re-deriving the finitized character of the unitary minimal models from its defining basis. This derivation turns out to be somewhat more economical than the original one using the path description [28]—although it is fair to stress that in its essence, it is not intrinsically different. But this analysis really serves as a verification of the correctness of the induced operator basis (i.e., the completeness of the set of restrictions). This makes this technique available for the investigation of more complicated problems (cf the concluding remarks).

The operator construction presented here for the $\mathcal{M}(k + 1, k + 2)$ models can be extended rather directly to the finite $\mathcal{M}(k + 1, 2k + 3)$ models. This special class of non-unitary models has been shown in [21] to have a path description very similar to that of the $\mathcal{M}(k + 1, k + 2)$ models except that the lattice is half-integer and peaks are forced to be at integer (x, y) positions. But the crucial point is that the weight function is defined exactly as for the unitary models: all vertices but the extrema contribute to $x/2$. The operator basis is thus very similar to that for the unitary models. It is briefly described in section 3. Note that the corresponding dual models are the graded \mathcal{Z}_k parafermions [8, 19].

Finally, we should point out that a somewhat analogous operator construction for the unitary minimal models has been considered before, in [11]. However, the main difference is that these authors have considered uncharged fermionic operators, which makes the basic relations somewhat different (in particular the partial charge condition below in (15) is more elaborated here). In addition, we claim to have an operator basis, hence no unphysical sequences get generated that would necessitate the introduction of projectors as in [11].

2. Paths for $\mathcal{M}(k + 1, k + 2)$ models and their operator interpretation

2.1. Paths, charge and the fermionic character

Let us start by recalling the definition of the paths representing the $\mathcal{M}(k + 1, k + 2)$ models. These are defined on an integer lattice in the first quadrant of the (x, y) plane and within the rectangle $0 \leq y \leq k$ and $0 \leq x \leq L$. An edge from x to $x + 1$ is either NE or SE. The *weight* w of a path is

$$w = \sum_{x=1}^{L-1} w(x), \quad \text{where} \quad w(x) = \frac{x}{4} |y_{x+1} - y_{x-1}|. \quad (1)$$

In other words, $w(x) = 0$ if x corresponds to the horizontal coordinate of an extremum of the path and it is equal to $x/2$ otherwise.

For a unique identification between the set of paths with specific boundary conditions and the states of a given module, we also need to specify the last edge (SE or NE) [26, 28]. Here it will be understood that all paths terminate with a SE edge. The generating function of paths with boundary conditions (y_0, y_L) yields the finitized character $\chi_{r,s}^{(L)}(q)$ with

$$r = y_L + 1 \quad \text{and} \quad s = y_0 + 1. \quad (2)$$

The condition $0 \leq y_L \leq k - 1$, which is forced by the SE edge termination, implies the right bound for r .³

For simplicity, we mainly confine ourselves to paths starting and terminating at height 0, which describe the states in the finitized vacuum module. Indeed, our aim is essentially to dress the known fermionic formula with a new interpretation and for that it is best to avoid unnecessary complications induced by boundary effects. Therefore, in the rest of this section

³ If we insist instead on terminating the path with a NE edge, the proper identification of the first label is $r = y_L$.

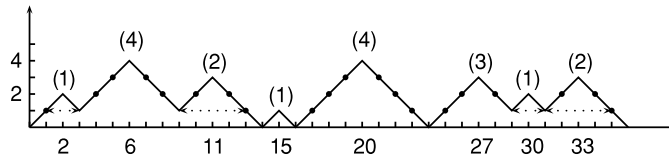


Figure 1. An example of a path valid for $k \geq 4$ starting at $y_0 = 0$ and ending at $y_{36} = 0$. There are four complexes in the path, delimited by the x positions 0, 14, 16, 24 and 36. The charge of each peak is indicated in parenthesis. A dotted line indicates the line from which the height must be measured to give the charge. The charge content is thus $n_1 = 3, n_2 = 2, n_3 = 1$ and $n_4 = 2$. The total charge is $\sum_i i n_i = 18$, half the length of the path. The weight of this path is half the sum of the x -coordinate of the dotted vertices ($w = 180$).

(except in the last subsection where the other modules are briefly considered), a path means a path with the particular condition $(y_0, y_L) = (0, 0)$.

A path is thus a sequence of peaks of height between 1 and k . The height by itself is not a good intrinsic characterization of the peaks. The proper concept is that of relative height [7] or charge [28]. We will use the latter qualitative. The definition is as follow. The charge of a peak with coordinates (x, y) is the largest number c such that we can find two vertices $(x', y - c)$ and $(x'', y - c)$ on the path with $x' < x < x''$ and such that between these two vertices there is no peak of height larger than y and every peak of height equal to y is located at its right [6].

For an isolated peak described by a triangle of height ℓ starting and ending on the x -axis, the charge is equal to the height. This is not as simple within a complex [7, 28], that is, within portions of the paths delimited by two points on the horizontal axis and containing more than one peak. In particular, if there are two peaks at the same height, the above definition indicates that it is the left-most peak which is attributed the largest charge, the one given by the height. In general thus, the charge of a peak is \leq its height. Figure 1 illustrates the definition of the charge.

A path is fully determined by the specification of its peak positions and their charges. In particular, the (vacuum) ground-state path for all unitary minimal models is described by a sequence of $L/2$ peaks of charge 1—see figure 2(a).

Denote by n_j the number of peaks of charge j . The length of a path is twice the sum of the charge of all its peaks

$$L = \sum_{j=1}^k 2jn_j. \tag{3}$$

Note that L is always even.

The values of n_j are the very basic numbers that enters in the fermionic expression of the characters. This expression, in its finitized form, reads [28]

$$\chi_{1,1}^{(L)}(q) = \sum_{n_1, \dots, n_k \geq 0} q^{nBn} \prod_{j=1}^{k-1} \begin{bmatrix} n_j + m_j \\ n_j \end{bmatrix}, \tag{4}$$

with

$$nBn = \sum_{i,j=2}^{k-1} n_i B_{ij} n_j, \quad \text{where } B_{ij} = B_{ji}, \quad B_{ij} = (i-1)j \quad \text{if } i \leq j, \tag{5}$$

and

$$m_j = 2n_{j+1} + 4n_{j+2} + \dots + 2(k-j)n_k. \tag{6}$$

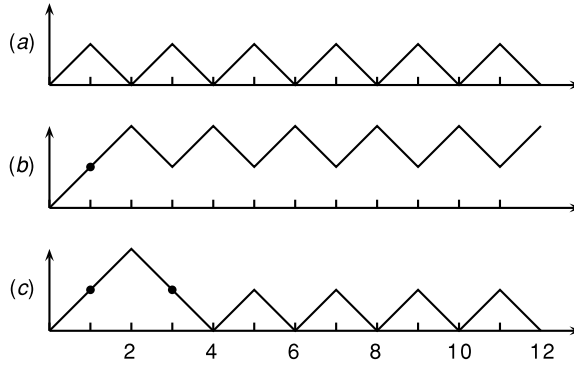


Figure 2. The ground state pertaining to the vacuum module of all the unitary minimal models is displayed in (a) for $L = 12$. The action of the operator b_1 on the ground state is presented in (b). This action is clearly non-local, modifying the path from the insertion point, $x = 1$, to the end, $x = 12$. One edge NE is created, linking $x = 1$ to $x = 2$, and the last SE edge of the original ground-state path is removed. The right boundary condition is clearly modified (the value of y_{12} being now 2 instead of 0). The action of the operator b_3^* on the path in (b) is given in (c). This action is again manifestly non-local: by creating a SE edge from $x = 3$ to $x = 4$, it modifies the whole path from the right of $x = 3$ and remove the final NE edge. The path illustrated in (c) is thus equivalent to the sequence $b_3^*b_1$ acting on the ground state configuration.

2.2. Operators acting on paths: setting the basis

The first objective is to describe any path by the successive action of operators acting on the ground-state configuration. We will thus interpret every weight-contributing vertex as resulting from the insertion of an operator. Next, we will determine, from the path characteristics, those conditions on sequences of these operators that capture the elements of an operator basis.

As already indicated in the introduction, this construction requires two types of operators, b and b^* , that create a NE and a SE edge respectively (the precise action is defined below). To these operators, we associate the respective charge $+1$ and -1 . Moreover, the mode of an operator corresponds to its horizontal insertion point. For instance b_5 indicates that the operator b acts at the vertex at horizontal position 5. The weight of an operator is half the value of its mode. The weight of a sequence of operators is thus the sum of the mode indices within the sequence divided by 2.

In order to define the action of these operators in a neat way, we observe that a path is determined by the sequence of its edges. With $NE \leftrightarrow 1$ and $SE \leftrightarrow -1$, it can thus be regarded as a succession of ± 1 :

$$\text{Path : } \alpha \equiv (\alpha_1, \alpha_2, \dots, \alpha_L), \quad \text{with } \alpha_i \in \{1, -1\} \quad \text{and} \quad \alpha_1 = -\alpha_L = 1. \quad (7)$$

The edge α_i links the vertices at x -position $i - 1$ and i . The restriction on the height takes the form

$$0 \leq h_i \leq k, \quad \text{where } h_i \equiv \sum_{j=1}^i \alpha_j. \quad (8)$$

The operator b_i acts on the vertex i , or equivalently, in-between the edges α_i and α_{i+1} . Its action is defined as

$$b_i(\alpha_1, \dots, \alpha_i, \alpha_{i+1}, \dots, \alpha_L) = (\alpha_1, \dots, \alpha_i, 1, \alpha_{i+1}, \dots, \alpha_{L-1}) \delta_{\alpha_i, 1} \delta_{\alpha_{i+1}, -1} \chi(h_i \leq k - 1), \quad (9)$$

where $\chi(a) = 1$ if a is true and zero otherwise. Let us rephrase this definition in words. Being designed to create a NE edge (which is the 1 in $i + 1$ -th position of the word), this operator b must act on a peak, i.e., in-between a pair of edges of type $(1, -1)$. This accounts for the two delta function. By creating a NE edge, this action of b manifestly affects the part of the path at the right of its insertion point. Its effect is as follows: it translates the remaining part of the path by one unit upward and one unit toward the right. Moreover, in order to be defined for L fixed, it removes the last edge. Finally, the restriction defined by the truth function takes care of the restriction (8). The action of the operator b_1 on the path representing the ground state is described in figure 2(b).

A number of immediate consequences follow from this definition. On any path α such that $b_i \alpha = \alpha' = (\alpha'_1, \dots, \alpha'_L) \neq 0$, then

$$\begin{aligned} (1) \quad & b_i^2 \alpha = 0, \\ (2) \quad & b_{i+2} b_i \alpha = 0 \\ (3) \quad & b_{i+k} \cdots b_{i+1} b_i \alpha = 0. \end{aligned} \tag{10}$$

Indeed, if $\alpha' \neq 0$, then $\alpha'_{i+1} = 1$ so that $b_i \alpha' \propto \delta_{\alpha'_{i+1}, -1} = 0$. Similarly, $\alpha'_{i+2} = -1$ so that $b_{i+2} \alpha' \propto \delta_{\alpha'_{i+2}, 1} = 0$. Finally, $b_{i+k} \cdots b_{i+1} \alpha' \propto \chi(h_i \leq 0) = 0$ since $\alpha' \neq 0$ implies that $h_i \geq 1$. These three relations capture the three constraints in the definition of the action of b_i specified by the two delta functions and the truth function.

After the action of ℓ operators of b -type, the boundary condition $y_L = 0$ is changed to $y_L = 2\lceil \ell/2 \rceil$. To preserve the original boundary condition, we need to act with a sequence of operators b^* such that each b^* creates a SE edge. Actually, for the end of the path to reach the horizontal axis, there must be as many b^* as b .

The precise definition of the action of the operator b_i^* is

$$b_i^*(\alpha_1, \dots, \alpha_i, \alpha_{i+1}, \dots, \alpha_L) = (\alpha_1, \dots, \alpha_i, -1, \alpha_{i+1}, \dots, \alpha_{L-1}) \delta_{\alpha_i, -1} \delta_{\alpha_{i+1}, 1} \chi(h_i \geq 1). \tag{11}$$

This implies

$$\begin{aligned} (1) \quad & b_i^{*2} \alpha = 0, \\ (2) \quad & b_{i+2}^* b_i^* \alpha = 0 \\ (3) \quad & b_{i+k}^* \cdots b_{i+1}^* b_i^* \alpha = 0. \end{aligned} \tag{12}$$

It also follows directly from the previous two definitions that

$$b_i b_i^* \alpha = b_i^* b_i \alpha = 0 \quad \text{and} \quad b_{i+1} b_i^* \alpha = b_{i+1}^* b_i \alpha = 0. \tag{13}$$

Let us now use these operators to set up a basis of states in the vacuum module which, in its final formulation, would be path-independent. For this we impose as constraints the three equalities in equations (10) and (12). Note that on the ground-state path of figure 2(a), the action of adjacent identical operators $b_j b_i$ or $b_j^* b_i^*$ must satisfy $j - i = 1 + 2n$ for n a positive integer (since the action is either successive, $j - i = 1$, or further separated by a sequence of peaks of charge 1, hence by an even number of x steps). For different operators, $b_j b_i^*$ or $b_j^* b_i$, the condition is rather $j - i = 2 + 2n$.

To these conditions, we need to impose a constraint that reflects the fact that the height of a vertex has to be everywhere $\leq k$ and always ≥ 0 . This has been partly taken into account by the condition (3) of equations (10) and (12). However, these conditions are not restrictive enough as they allow, for example, a sequence of ℓ ($\leq k - 1$) b , followed by a b^* and then a sequence of $\ell' b$, with $\ell + \ell' \geq k + 2$, which would lift a vertex to a height larger than k . Such possibilities need to be excluded. This forces the charge computed from the beginning to any point in the sequence of operators (the sequence being read from left to right) to lie within the range $[0, k - 1]$. This can be made more precise as follows. Consider any sequence of

modes of the form $c_{i_m} c_{i_{m-1}} \cdots c_{i_1}$, with c being either b or b^* , and denote the charge by q (i.e., $q(b) = -q(b^*) = 1$). Introduce the partial charge q_ℓ as

$$q_\ell = \sum_{n=1}^{\ell} q(c_{i_n}). \tag{14}$$

Then, to ensure that $0 \leq y \leq k$ everywhere, one requires all partial charges to satisfy

$$0 \leq q_\ell \leq k - 1 \quad (1 \leq \ell \leq m). \tag{15}$$

We stress that (15) embodies a non-locality aspect: it can be viewed as a constraint on the type of operator to-be-inserted which involves not only the preceding operator, but, in principle, all those which have been previously introduced⁴.

The set of conditions identified completely define the operator basis proper to the vacuum module. These conditions can be summarized as follows:

$$\begin{aligned} c_{i_m} \cdots c_{i_2} c_{i_1} : \quad & c \text{ is either } b \text{ or } b^*, \text{ with } q(b) = -q(b^*) = 1, \\ i_{s+1} - i_s = 2n + 1 + \frac{1}{2}(1 - q(c_{i_{s+1}})q(c_{i_s})), \\ & \text{with } i_1 \geq 1 \text{ and } i_m \leq L - 1, \\ 0 \leq q_\ell \leq k - 1, \quad & \text{with } q_1 = 1 \text{ and } q_m = 0. \end{aligned} \tag{16}$$

The finitized (vacuum) character is the generating function of all the sequences of operators subject to (16). Its construction is considered in the next two subsections. At the end, the correctness of the resulting expression testifies that of the basis (16).

2.3. Combinatorics of the operator basis: the minimal-weight configuration and its displacements

Our aim is now to derive the generating function of all the strings of operators satisfying the conditions (16). The first step is the identification of a convenient choice of summation variables. For paths, it is the charge content of the peaks, that is, the different n_j , which provides the natural summation variables. The analogous variables for strings of operators are readily identified.

Let us call a sequence of 2ℓ operators ordered (with decreasing value of their mode, i.e., position action) as $b^* \cdots b^* b \cdots b$ a ℓ -block. Denote by p_ℓ the number of ℓ -blocks. By analogy with paths, the operator content over which the summation will be performed is thus the block content, that is, the values of p_1, \dots, p_{k-1} .

The problem can thus be formulated more precisely as follows: (1) enumerate all possible sequences with a specified block content, and (2) sum over all block contents compatible with the finitization.

It is rather immediate to see that among all sequences of operators with a specific block content, the pattern with minimal weight is the one with all blocks unmixed⁵, as closely

⁴ The constraint on successive allowed modes (cf (3) in equations (10) and (12)) can be viewed as a sort of weak form of the exclusion principle that governs the quasi-particle parafermionic basis (as it appears in the form of the ‘difference 2 condition at distance $k - 1$ ’ [18, 25]). But the above specific bound on the charge of strings of b and b^* modes is a novelty with regard to constraints on the charge in parafermionic theory since the parafermionic charge—when used to organize the modules—is in fact defined modulo $2k$, so that no states are really forbidden by the requirement that the relative charge r should be reshuffled within the bounds $-k < r \leq k$. Viewed differently, the charge of a string of operators preceding the application of a given b or b^* acts as a sort of exclusion principle while in the parafermionic context, this is not so; it is true that the charge up to the point of application of a given parafermionic mode does play a role but only as a shift of the mode index.

⁵ The effect of mixing the blocks is considered in the next section and shown there to increase the weight with respect to the (unmixed) configuration considered here.

packed as possible, and ordered in decreasing values of ℓ (from right to left). Rephrased more explicitly, this state is described by sets of operators with modes as closely packed as allowed by the hard-core repulsion (16) and ordered as follows, from right to left to: the $p_{k-1}(k-1)$ -blocks, followed by the $p_{k-2}(k-2)$ -blocks, down to the p_1 blocks of type 1.

The rationale for identifying this particular operator sequence with the lowest weight state is that the larger the value of ℓ (the block type), the greater is the number of b operators contributing to the weight so that it is energetically favorable to have them in lowest available modes. Note that even in a closely-packed configuration, the ordering of the blocks matters significantly with regard to the total weight since there is a gap of 1 in-between every pair of adjacent bb^* and b^*b .

Let us now determine the weight of the minimal-weight configuration. Consider the closely-packed ℓ -block of the form $b_{i_0+2\ell+1}^* \cdots b_{i_0+\ell+2}^* b_{i_0+\ell} \cdots b_{i_0+1}$. Its total weight is

$$\frac{1}{2} \sum_{j=1}^{2\ell+1} (i_0 + j) - \frac{1}{2} (i_0 + \ell + 1) = \ell(i_0 + \ell + 1), \tag{17}$$

where the subtraction keeps track of the gap of 1 between $b_{i_0+\ell+2}^*$ and $b_{i_0+\ell}$. For a sequence of p_ℓ such blocks, one has

$$\frac{1}{2} \sum_{j=1}^{2(\ell+1)p_\ell} (i_0 + j) - \frac{1}{2} \sum_{j=1}^{2p_\ell} (i_0 + j(\ell + 1)) = \ell p_\ell (i_0 + (\ell + 1)p_\ell). \tag{18}$$

This clearly indicates that the largest the value of ℓ , the more energetic it is to translate ℓ -blocks. The value of i_0 is the sum of the diameter of all higher blocks in their closely-packed form:

$$i_0 = 2 \sum_{j=\ell+1}^{k-1} (j + 1)p_j. \tag{19}$$

The minimal-weight configuration with specified block content (p_1, \dots, p_{k-1}) is thus found to be

$$\sum_{i,j=1}^{k-1} p_i [\min(i, j)(\max(i, j) + 1)] = \sum_{i,j=1}^{k-1} p_i B_{i+1,j+1} p_j, \tag{20}$$

where B_{ij} is defined in (5).

Consider now all possible translations of this configuration. By this we mean all possible displacements of the various operators which preserve their ordering. The displacement of each operator is measured from its position in the minimal-weight configuration. Consider first the leftmost operator, which is of type b^* . Its mode is $M - 1$, with M defined by

$$M = 2 \sum_{j=1}^{k-1} (j + 1)p_j. \tag{21}$$

M is the sum of the diameter of all the blocks. The operator b_{M-1}^* can be translated, in units of 2, up to $L - 1$. Its maximal displacement is thus $L - M$ and the weight increase is $(L - M)/2$.

The penultimate operator can similarly be translated by an even integer which is not larger than the displacement of the last operator. More generally, the different displacements of the various operators, taken from left to right, are in correspondence with a sequence of numbers, which, by construction (since the ordering of the operator must remain unaffected) are non-increasing. Denote these different numbers by $(2\mu_1, \dots, 2\mu_p)$, where $2\mu_i$ is the displacements

of the i th operator counted from the left, so that $\mu_i \geq \mu_{i+1}$ and $\mu_1 \leq (L - M)/2$. P is the total number of operators:

$$P = 2 \sum_{j=1}^{k-1} j p_j. \tag{22}$$

Note that some of these μ_j (necessarily those at the end of the sequence) are allowed to vanish.

We now argue that the weight increase resulting from such displacements is

$$n = \mu_1 + \dots + \mu_P. \tag{23}$$

Consider first the weight increase caused by the displacement of the last operator. A shift by $2\mu_1$, i.e.

$$b_{M-1}^* \dots \rightarrow b_{M-1+2\mu_1}^* \dots, \tag{24}$$

clearly increases the weight by μ_1 . Consider now a shift of the penultimate operator. Since it starts from a closely-packed configuration and that the ordering needs to be preserved, a shift of $2\mu_2$ of this operator must be accompanied by a shift of the same amount of the last operator by $2\mu_2$. The weight increase is $2\mu_2$. If the last operator is then further shifted by $2(\mu_1 - \mu_2)$, augmenting thereby the weight by $\mu_1 - \mu_2$, the total weight increase is $\mu_1 + \mu_2$. By iterating this argument, we arrive at the claimed result.

Counting the possible displacements, while keeping track of the corresponding weight increase, amounts to evaluating the following multiple summation:

$$\sum_{\mu_1=0}^{(L-M)/2} \sum_{\mu_2=0}^{\mu_1} \dots \sum_{\mu_P=0}^{\mu_{P-1}} q^{\mu_1+\dots+\mu_P}. \tag{25}$$

This is equivalent to enumerate the partitions (μ_1, \dots, μ_P) of $n = \mu_1 + \dots + \mu_P$ into at most P parts, each part being at most equal to $(L - M)/2$ and with each partition being weighted by q^n . The generating function for the number of partitions $p(r, m, n)$ of n into at most m parts each $\leq r$ is (cf [1] theorem 3.1):

$$\sum_{n \geq 0} p(r, m, n) q^n = \begin{bmatrix} r + m \\ m \end{bmatrix}, \tag{26}$$

where the q -binomial coefficient is

$$\begin{bmatrix} a \\ b \end{bmatrix} = \frac{(q)_a}{(q)_{a-b}(q)_b}, \tag{27}$$

and with the q -factorial function $(q)_a$ being defined as

$$(q)_a = (1 - q) \dots (1 - q^a). \tag{28}$$

One thus concludes that

$$\sum_{\mu_1=0}^{(L-M)/2} \sum_{\mu_2=0}^{\mu_1} \dots \sum_{\mu_P=0}^{\mu_{P-1}} q^{\mu_1+\dots+\mu_P} = \begin{bmatrix} L/2 - M/2 + P \\ P \end{bmatrix}. \tag{29}$$

Let us define the integer p_0 as $L - M = 2p_0$ (so that p_0 is the maximal possible value of μ_1). We can thus write

$$L = 2 \sum_{j=0}^{k-1} (j + 1) p_j. \tag{30}$$

Then the q -binomial reads

$$\begin{bmatrix} L/2 - M/2 + P \\ P \end{bmatrix} = \begin{bmatrix} p_0 + P \\ P \end{bmatrix}. \tag{31}$$

2.4. Combinatorics of the operator basis: mixing the blocks

Up to this point we have only considered those displacements that preserve the ordering of the operators as determined by the minimal-weight configuration. The rationale for maintaining the ordering is the hard-core repulsion between the individual operators (cf (16)). However, this leaves open the possibility of moving an operator through a charge-less combination, that is, through a block. Equivalently phrased, we must keep track of the possibility of inserting small blocks within larger ones.

The operation of mixing the blocks while maintaining the block content raises the issue of determining the block content of a generic sequence of operators. This can be done simply, by identifying the blocks successively, from the smallest to the largest ones. One thus first identifies all the 1-blocks, that is, all the pairs b^*b within subsequences of the form $b^i b^* b b^{*j}$, with $\max(i, j) > 0$. Remove all these pairs; their number is the value of p_1 for the sequence under study. Proceed then similarly to the identification of the 2-blocks. Iterate the procedure until all $(k - 2)$ -blocks have been identified. The remaining operators form $(k - 1)$ -blocks. For instance (assuming the modes to be increasing from right to left), for the sequence

$$b^*b^*b^*b^*b^*bbbbb^*bb^*b^*b^*bbb^*b^*bbb^*bbb, \quad (32)$$

we have the following successive pairings (where the subscript indicates the content of the enclosed block, hence the step at which the corresponding grouping is performed):

$$b^*b^*(b^*b^*b^*bbb)_3(b^*b)_1b^*b^*b^*bbb(b^*b^*bb)_2(b^*b)_1bbb. \quad (33)$$

The remaining operators form a 6-block. The complete content of this sequence is $p_1 = 2$, $p_2 = p_3 = p_6 = 1$.

The first step in the analysis of mixing the blocks is to identify all possible (but distinct) ways of inserting a j -block within a ℓ -block, for $j < \ell$. It is understood that the block content (which at this point is still fixed) must be preserved by this procedure. In this regard, what has to be prevented is the creation, from the mere mixing, of blocks larger than ℓ . This is most conveniently ensured by imposing a constraint on the partial charges. For every insertion, we need to verify that the partial charge satisfies (in this context) $0 \leq q_n \leq \ell$, with $1 \leq n \leq 2(j + \ell)$.

Before turning to the general case, it is convenient to treat a simple example, say $(j, \ell) = (1, 2)$. One starts with $(b^*b)_1(b^*b^*bb)_2$ and move the pair b^*b within the 2-block. The possible cases, including the original configuration, are

$$\begin{aligned} (b^*b)b^*b^*bb, & & b^*(b^*b)b^*bb, & & b^*b^*(b^*b)bb, \\ b^*b^*b(b^*b)b, & & b^*b^*bb(b^*b), & & \end{aligned} \quad (34)$$

(with the original displaced pair being identified by parentheses). The third configuration has to be excluded since q_3 is 3, which is not within the range allowed here, namely $[0, 2]$. (Equivalently, its decomposition in blocks does not preserve the block content $p_1 = p_2 = 1$; it rather corresponds to $p_1 = p_2 = 0$, $p_3 = 1$.) Removing the parentheses, we see that the second and the fourth cases have the same structure. The corresponding pattern must be counted only once. We are left with three possibilities:

$$b^*bb^*b^*bb, & & b^*b^*bb^*bb, & & b^*b^*bbb^*b. \quad (35)$$

Inserting the modes appropriate for the closely-packed version yields

$$b_9^*b_7b_5^*b_4^*b_2b_1, & & b_9^*b_8^*b_6b_4^*b_2b_1, & & b_9^*b_8^*b_6b_5b_3^*b_1. \quad (36)$$

The respective weights are 14, 15 and 16. In other words, the distinct insertions of a 1-block b^*b within a 2-block modify the weight by steps of 1 and there are three distinct

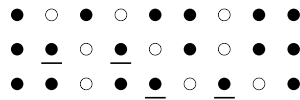


Figure 3. A pictorial representation of the three configurations corresponding to the mixing of a 1-block into a 2-block.

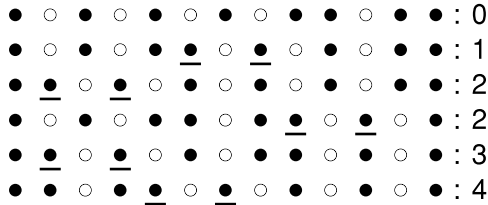


Figure 4. The configurations describing the mixing of a 1-block into two 2-blocks.

configurations. With the weight evaluated with respect to the minimal-weight configuration, the three possibilities are taken care of by the q -binomial

$$\begin{bmatrix} 3 \\ 1 \end{bmatrix} = 1 + q + q^2. \tag{37}$$

In order to see easily that the weight is increased by 1 for each successive displacements of the pair b^*b within the 2-block, let us introduce the following pictorial devise. Indicate by \bullet an occupied mode and by \circ an empty one—a hole—(representing the gap in-between a bb^* or a b^*b pair). For the three configurations in (36), we have the representations given in figure 3.

In this representation, a configuration is obtained from the previous one by the displacement toward the left of two dots separated by a hole. These displaced dots must be free to move together by one unit and their displacement should not generate a configuration containing three successive dots—which would necessarily violate the charge constraint since no gaps between filled points mean that they are all occupied by the same type of operators. The dots which have been moved are underlined. From this pictorial description, it is manifest that the weight is increased by 1 at each step.

As another example, consider $p_1 = 2$ and $p_2 = 1$. We have the six distinct configurations displayed in figure 4 (where again the displaced dots are underlined), together with their relative weight at their right. From the second configuration (of relative weight 1), there are two ways of displacing two dots by one unit and this leads to the two configurations of relative weight 2. These different possibilities are captured by the factor

$$\begin{bmatrix} 4 \\ 2 \end{bmatrix} = 1 + q + 2q^2 + q^3 + q^4. \tag{38}$$

With p_1 1-blocks and p_2 2-blocks, the proper q -binomial factor takes the form

$$\begin{bmatrix} p_1 + 2p_2 \\ p_1 \end{bmatrix}. \tag{39}$$

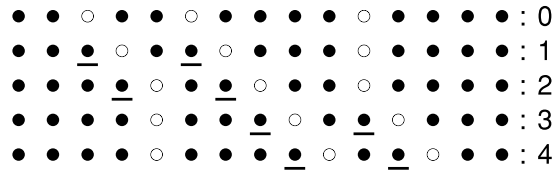


Figure 5. The five configurations associated with the mixing of a 2-block into a 4-block.

Let us now turn to the analysis of the more general situation where a j -block is inserted within a ℓ -block. The insertions within the substring $b^{*\ell}$ lead to configurations of the form

$$b^{*i}(b^{*j}b^j)b^{*\ell-i}b^\ell. \tag{40}$$

The partial charge $q_{2\ell-i+j}$ (computed as usual from right to left) is $j + i$; in order to respect the bound $0 \leq q_n \leq \ell$, we need to have $i + j \leq \ell$. This excludes the insertion at the center of the ℓ -block ($i = \ell$) and its vicinity. There are thus $\ell - j + 1$ distinct insertions within the substring $b^{*\ell}$, counting the one with $i = 0$ (which is the original configuration where the j -block follows the ℓ -block). Within the substring b^ℓ , there are also $\ell - j + 1$ possible insertions, but one of these is identical to one already considered in (40) and it should not be counted twice. The total number of distinct cases is $2(\ell - j) + 1$. Each unit displacement of the j -block within the ℓ -block increases the weight by 1. The counting of possibilities, taking the weight change into account, is thus

$$\begin{bmatrix} 1 + 2(\ell - j) \\ 1 \end{bmatrix}. \tag{41}$$

This combinatorial factor and the weight shift are nicely exemplified by our pictorial representation. In the case where a j -block is inserted within a ℓ -block, we displace the leftmost pair of dots that are separated by j units ($j - 1$ dots and 1 hole) and which are free to move simultaneously without generating a sequence of $\ell + 1$ successive dots. The displacements are toward the left, by one unit each time. For $j = 2$ and $\ell = 4$, this leads to the configurations shown in figure 5. The moved dots are underlined. To reach the fourth configuration, we observe that the leftmost holes cannot be filled due to the charge constraint. In the operator language, this means that we need to switch from the resulting configuration to its symmetrical counterpart and start the displacements from there on. In other words, in our pictorial representation, the next move involves the filling of the next two holes.

The description of the various distinct configurations obtained from mixing the blocks in terms of the successive displacements (of one unit toward the left) of two dots separated by j units readily implies that the weight increases by 1 at each step.

The combinatorial factor with $p_j j$ -blocks and $p_\ell \ell$ -blocks is the q -deformation of the binomial coefficient that counts the number of ways the $p_j j$ -blocks can be inserted within the larger ones. This is

$$\begin{bmatrix} p_j + 2(\ell - j)p_\ell \\ p_j \end{bmatrix}. \tag{42}$$

The generalization to a generic sequence of blocks is

$$\prod_{j=1}^{k-2} \begin{bmatrix} p_j + 2p_{j+1} + 4p_{j+2} + \dots + 2(k - 1 - j)p_{k-1} \\ p_j \end{bmatrix} \equiv \prod_{j=1}^{k-2} \begin{bmatrix} p_j + \tilde{p}_j \\ p_j \end{bmatrix}, \tag{43}$$

with

$$\tilde{p}_j = 2p_{j+1} + 4p_{j+2} + \dots + 2(k - 1 - j)p_{k-1}. \tag{44}$$

Note that if we extend the notation \tilde{p}_j to the case where the index is 0, we have

$$\tilde{p}_0 = 2p_1 + 4p_2 + \dots + 2(k-1)p_{k-1} = P, \tag{45}$$

where P is the number of operator defined in (22).

2.5. The generating function

We have constructed all the pieces that compose the generating function for the string of operators subject to the rules (16). We now put them together. The generating function for all the possible displacements of the minimal-weight configuration, with a given block content and the ordering of the operators maintained fixed, is

$$q^{pB'p} \begin{bmatrix} p_0 + \tilde{p}_0 \\ p_0 \end{bmatrix}, \tag{46}$$

where

$$pB'p = \sum_{i,j=1}^{k-1} p_i B_{i+1,j+1} p_j. \tag{47}$$

We next have to consider all possible mixing of operators that preserve the block content. With the weight evaluated relative to that of the minimal-weight configuration, this amounts to add the factor (43). The resulting expression is (note the lower bound in the product)

$$q^{pB'p} \prod_{j=0}^{k-2} \begin{bmatrix} p_j + \tilde{p}_j \\ p_j \end{bmatrix}, \tag{48}$$

with \tilde{p}_j defined in (44). This is the generating function for all strings of operators with given values of p_1, \dots, p_{k-1} . The full generating function, for fixed length, is obtained by summing over all p_1, \dots, p_{k-1} compatible with (30):

$$G^{(L)}(q) = \sum_{p_1, \dots, p_{k-1} \geq 0} q^{pB'p} \prod_{j=0}^{k-2} \begin{bmatrix} p_j + \tilde{p}_j \\ p_j \end{bmatrix}. \tag{49}$$

This is the finitized vacuum character for the $\mathcal{M}(k+1, k+2)$ minimal model. Note that the finitization is encoded in the q -binomial with $j = 0$. With $p_j = n_{j+1}$ and $\tilde{p}_j = m_{j+1}$, we recover expression (4).

The infinite length limit is taken by setting $p_0 \rightarrow \infty$. For this we use

$$\lim_{n \rightarrow \infty} \begin{bmatrix} n \\ m \end{bmatrix} = \frac{1}{(q)_m}, \tag{50}$$

so that

$$G^{(\infty)}(q) = \chi_{1,1}(q) = \sum_{p_1, \dots, p_{k-1} \geq 0} \frac{q^{pB'p}}{(q)^{\tilde{p}_0}} \prod_{j=1}^{k-2} \begin{bmatrix} p_j + \tilde{p}_j \\ p_j \end{bmatrix}. \tag{51}$$

2.6. Modules other than the vacuum one

As already said, the various irreducible modules labeled by the pair (r, s) are characterized by their boundary conditions (2) (assuming a final SE edge). Note that the module (r, s) is identified with the one labeled by $(k+1-r, k+2-s)$. Using this equivalence, it is always possible to choose $s \leq r$, that is, $y_0 \leq y_L$. The ground state in the case $r = s$ is similar to that for the vacuum module: it zigzags between the height $s - 1$ and s . From this ground

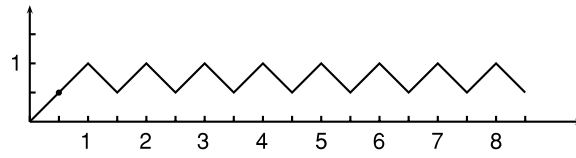


Figure 6. The ground-state path in the vacuum module of all $\mathcal{M}(k + 1, 2k + 3)$ models. The single vertex contributing to the weight, which is $1/4$, is indicated by a dot.

state, one can reach the ground state for all the modules with $r > s$ by acting with the string $b_{r-s} \cdots b_1$. The complete set of states in the finitized (r, s) module with $r \geq s$ is obtained by adjoining to this initial string a charge-less sequence satisfying conditions (16) except that the bound on i_1 is modified to read $i_1 \geq (r - s + 1)$ and the partial charge of the added sequence will now satisfy $-(r - s) \leq q_\ell \leq k - 1 - r + s$. That different modules ($r \geq s$) can be described in terms of a single one ($r = s$) testifies a sort of economy of the operator description. However, the weights of the modules so related cannot be read off directly from this description: $h_{r,s} \neq h_{s,s} + 1/2 + \cdots + (r - s)/2$. (The exception is the Ising model when $s = 1$.) We will not detail the combinatorial analysis of this more general case as it does not significantly differs from that already considered and that no new formulae are so generated.

3. Operator description of the $\mathcal{M}(k + 1, 2k + 3)$ models

3.1. Paths and operators for the $\mathcal{M}(k + 1, 2k + 3)$ models

It is quite remarkable that the $\mathcal{M}(k + 1, 2k + 3)$ models do have a path description very similar to that of the $\mathcal{M}(k + 1, k + 2)$ models [21]. The essential difference is that the lattice is half-integer and peaks (whose charge can range from $1/2$ to $k - 1/2$) are forced to be at integer (x, y) positions. Here again the paths are defined within the rectangle $0 \leq y \leq k$ and $0 \leq x \leq L$ (but the lattice delimited by this rectangle is now half-integer). Moreover, the weight function is defined exactly as for the unitary models: all vertices but the extrema contribute to $x/2$. The paths start at integer height y_0 and end up at half-integer height y_L , with a SE edge (where here an edge connects the points (x, y) and $(x + 1/2, y \pm 1/2)$). The module labels are related to the boundary values as

$$r = y_L + \frac{1}{2} \quad \text{and} \quad s = 2y_0 + 1. \tag{52}$$

The ground state pertaining to the vacuum module of all the models of this class is displayed in figure 6. Observe that it is necessary to have a peak of charge 1 preceding those of charge $1/2$. An example of path, appropriate to any model with $k \geq 2$, is displayed in figure 7.

Like for the unitary case, the path can be dressed with a natural operator interpretation, in terms of the operators b and b^* defined as before (although their modes are now multiple of $1/2$). Note however that the ground-state path cannot be regarded as resulting from the action of $b_{1/2}$ on the path that would zigzag between 0 and $1/2$ since such a path is not allowed, its peaks lying at half-integer positions. Therefore, we consider this $b_{1/2}$ to be built-in the vacuum definition. The partial charge (which is again computed from right to left) in this case satisfies

$$-1 \leq q_\ell \leq 2(k - 1). \tag{53}$$

For instance, to the path of figure 7, we associate the sequence: $b_{34/2}^* b_{33/2}^* b_{31/2} b_{30/2} b_{27/2} \cdots b_{4/2}$.

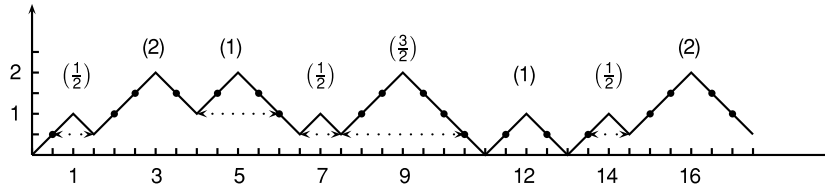


Figure 7. An example of a $\mathcal{M}(k + 1, 2k + 3)$ path for $k \geq 2$ starting at $y_0 = 0$ and ending at $y_{35/2} = 1/2$. Weight contributing vertices are indicated by dots. Note that the (x, y) coordinates of the peaks are integers. The charge of the peaks is determined as previously but with the path augmented by a final SE edge. The charge is the height measured from the indicated dotted line or the horizontal axis and it is indicated above each peak.

Our task is again to write down the basis pertaining to these models. But to motivate some of the differences with the $\mathcal{M}(k + 1, k + 2)$ basis, it is convenient to consider briefly the first model ($k = 1$) of the sequence, focusing again on the simple vacuum boundary conditions $(y_0, y_L) = (0, 1/2)$. Appropriate states are thereby described by charge-less sequences of operators.

3.2. The $\mathcal{M}(2, 5)$ model

For $k = 1$, the only allowed operator combinations are of type bb^* . Moreover, these pairs must necessarily occur in closely-packed doublets, in the form $b_{i+1}b_i^*$ because they cannot be separated by peaks of charge $1/2$ since these would then necessarily create peaks at non-integer positions.

Let p be the number of 1-block of the type bb^* . The minimal-weight configuration with p such pairs is $b_{3/2+2p-1} \cdots b_{7/2}^* b_{5/2} b_{3/2}^*$ and its weight is easily found to be $p^2 + p$. Displacements need to be considered but each displacement must involve the two members of the block. Both mode indices change then by steps of 1, so that the weight change for every block displacement is $(1 + 1)/2 = 1$. The maximal displacement of the leftmost b is from $3/2+2p-1$ to $L-1$ (this last possible position being determined by the constraint of a final SE edge). The maximal displacement is thus $L - 3/2 - 2p$ and since the two parts of the doublet are moved together, this is the weight shift. Since there are p blocks, one has to q -enumerate the number of partitions into at most p parts with the maximal part $\leq L - 3/2 - 2p$. This is given by the combinatorial factor

$$\left[\begin{matrix} L - 2p - 3/2 + p \\ p \end{matrix} \right] = \left[\begin{matrix} L - p - 3/2 \\ p \end{matrix} \right]. \tag{54}$$

The generating function reads

$$\sum_{p \geq 0} q^{p^2+p} \left[\begin{matrix} L - 2p - 3/2 \\ p \end{matrix} \right] \xrightarrow{L \rightarrow \infty} \sum_{p \geq 0} \frac{q^{p^2+p}}{(q)_p}. \tag{55}$$

This is the expected result. To compare with the expressions presented in [21], note that $p = n_1 - 1$, where n_1 is the number of peaks of charge 1.

3.3. The $\mathcal{M}(k + 1, 2k + 3)$ operator basis

For the $\mathcal{M}(3, 7)$ model, the only allowed operator combinations are of type bb^*b^2 , b^*b^2 or bb^* , which we will call 3-, 2- and 1-blocks respectively. In the general case, one needs to consider ℓ -blocks with $1 \leq \ell \leq 2k - 1$, with the following structure:

$$\begin{aligned} &bb^{*\ell}b^{\ell-1} && \text{for } \ell \text{ odd} \\ &b^{*\ell}b^\ell && \text{for } \ell \text{ even.} \end{aligned} \tag{56}$$

The strategy for constructing the generating function is again to identify the minimal-weight configuration for a fixed block content, determine its weight, q -enumerate all possible configurations, and sum over the block content.

The problem of main interest in this novel context is to unravel the conditions to be imposed on successive operators which ensure the peaks, in the associated path representation, to lie at integer positions. As a first orientation on this question, consider the insertion of a 1-block within a 3-block. The allowed possibilities are the following:

$$(bb^*)bb^*b^*b^*bb, \quad bb^*(bb^*)b^*b^*bb, \quad bb^*b^*b^*(bb^*)bb, \quad bb^*b^*b^*bb(bb^*), \tag{57}$$

(i.e., the 1-block is moved by two units in each step) while those that must be discarded are

$$b(bb^*)b^*b^*b^*bb, \quad bb^*b^*(bb^*)b^*bb, \quad bb^*b^*b^*b(bb^*)b. \tag{58}$$

The first discarded one does not satisfies the bound (53) (i.e., $q_6 = -2$). The other two are not allowed because they have a peak at half-integer position.

It is clear from this example that to ensure the integrality of the peak position, which occurs in-between a b and a b^* , one must impose a condition formulated in terms of the parity of the partial charge up to and including the operator b just before the b^* to be inserted. This condition is simply that the partial charge before the insertion of b^* needs to be even:

$$\dots b_{i_{\ell+1}}^* b_{i_\ell} \dots : \quad q_\ell = 2n. \tag{59}$$

Again we stress that the computation of the partial charge must not take into account the operator $b_{1/2}$ which is always present and considered to be part of the mere definition of the vacuum.

Finally, there is also a potential constraint on the modes of adjacent operators that generalize the closely-packed condition of the pairs bb^* in the $\mathcal{M}(2, 5)$ model. With c being either b or b^* , the generalized form of this condition is the following: to the usual hard-core repulsion

$$\dots c_{i_{\ell+1}} c_{i_\ell} \dots : \quad i_{\ell+1} - i_\ell = n + \frac{1}{2} + \frac{1}{4}[1 - q(c_{i_\ell})q(c_{i_{\ell+1}})] \tag{60}$$

one adds the condition

$$n = 0 \quad \text{if } q_\ell = 2m + 1. \tag{61}$$

These conditions (59), (60) and (61) are the essential new characteristics defining the basis as compared to the unitary one.

This completes our presentation of the operator basis for the $\mathcal{M}(k + 1, 2k + 3)$ models. The combinatorial analysis of these models, in terms of paths, is presented in [21] and it will not be rephrased in the operator basis. Again, the resulting character formulae are the same.

4. Concluding remarks

We have presented a rather natural operator description of the states in the irreducible modules of the unitary minimal models. This fermionic-type operator construction is derived from the

RSOS path representation of the states. On the paths, these operators act in a non-local way. This construction can be abstracted and made independent of the path used for their definition. This leads to the formulation of the set of conditions (16) that define a basis. The correctness of these basis conditions has been supported by the derivation (relying solely on (16)) of the generating function that reproduce the known fermionic characters [28]. In this abstract form, an element of non-locality is still present through a condition involving the partial charge of a sequence of operators.

The operational description is somewhat economical when compared to the path in that we do not have to take into account peaks of charge 1 which do not contribute to the weight. This is similar to the relation between Bressoud and RSOS paths in the description of the \mathcal{Z}_k parafermions [20]: the variables in the former case are peaks of charge between 1 and $k - 1$ while peaks of charge 1 up to k are considered in the RSOS case. Although peaks of charge k do not contribute to the weight, they allow for a natural way of finitizing the paths. In contrast, the Bressoud paths are not naturally finitized. Similarly, the minimal models have a natural finitization in their RSOS path formulation, while the operator approach does not entail such a natural finite form.

Let us expand a bit on the issue of finitization. The way we have finitized the characters in the operator basis by constraining the maximal value of the rightmost operator is of course dictated by the path representation that constitute the starting point of the construction. But once the basis is written and taken from an intrinsic standpoint, outside its original path context, the consideration of a finite length becomes rather artificial. It is thus more natural to set $L \rightarrow \infty$ in (16). The generating function of the resulting basis elements leads then directly to the conformal character. In this regard, it is appropriate to recall one of the main reasons that makes finitized expressions interesting: finitization offers a powerful way of proving the correctness of conjectured fermionic expressions by demonstrating that they satisfy the same recurrence relations that characterize the finitized configuration sums (see e.g., [4, 26]). But this motivation becomes much less striking within the context of a constructive method such as that presented here (and [28])⁶.

When considered from the point of view of the general Forrester–Baxter RSOS paths, the operator method presented here applies only to the unitary minimal models. The crucial simplifying feature of the unitary models is that the weight of the contributing vertices (those with non-zero weight) depends solely upon their x -position. However, we have shown in [21] that the $\mathcal{M}(k + 1, k + 2)$ Bailey duals [13], the $\mathcal{M}(k + 1, 2k + 3)$ models, do have a similar path description (albeit without yet a RSOS underlying construction). It is thus not surprising to find that our operator setting can be lifted to that case.

But what is probably more compelling (and which, ultimately, might be viewed as the best immediate argument justifying the interest of this new approach) is that this operator construction can be extended to the superconformal unitary models. For these models, the constructive method following the lines of [28] is not so directly worked out. It turns out that in this context, the logic underlying the fermi-gas description of the paths is reversed: a path has a natural operator construction and it is this very operator interpretation which, once finitized, allows us to clearly identify the particles within the path [22]. We suspect that the same situation will hold for the fermi-gas analysis of the more general models [9] $\widehat{su}(2)_k \otimes \widehat{su}(2)_\ell / \widehat{su}(2)_{k+\ell}$, for $\ell > 2$: working at the level of the operator basis induced by the path description is likely to be simpler than working directly at the level of the paths.

⁶ Apart from their intrinsic interest, the other motivation for finitized characters has already pointed out in the introduction, which is the possibility of defining dual characters from the $q \rightarrow q^{-1}$ transformation [5, 9, 12, 28].

There are a number of natural extensions of this work. At first, there is a major loose end from the point of view of our motivating vague discussion in the introduction, which led us to suspect the operator basis to be non-local and potentially related to parafermionic-type quasi-particles. Clearly, the whole conformal-field-theoretical interpretation of these operators still has to be unravelled. Finally, it would be of interest to see if this new operator could be used to understand the action of the Virasoro algebra on the paths, in the spirit of previous works (see e.g. [27] and references therein).

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